Transfinite mean value interpolation

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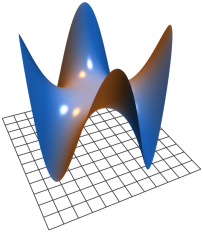

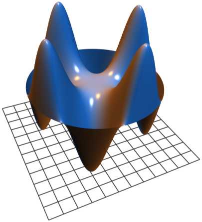

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Transfinite interpolation

Given $\Omega \subset \mathbb{R}^2$, a convex or non-convex set, possibly with holes.

Lagrange transfinite interpolation
We are given $f : \partial\Omega \to \mathbb{R}$.
Find $g : \Omega \to \mathbb{R}$ that interpolates f on $\partial\Omega$.

Hermite transfinite interpolation
We are given $f : \partial\Omega \to \mathbb{R}$ and $\frac{\partial f}{\partial \mathbf{n}} : \partial\Omega \to \mathbb{R}$.
Find $g : \Omega \to \mathbb{R}$ that interpolates f and $\frac{\partial g}{\partial \mathbf{n}}$ matches $\frac{\partial f}{\partial \mathbf{n}}$ on $\partial\Omega$.

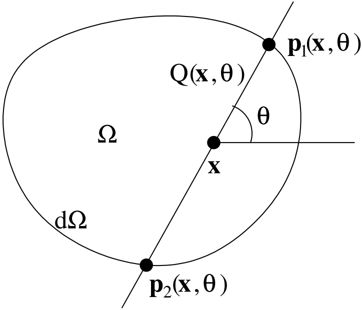
- Lagrange can be solved by solving the harmonic equation
- Hermite can be solved by solving the biharmonic equation

$\Longrightarrow$ But we want something simpler...

Pseudo-harmonic interpolation [Gordon, Wixom 1974]

Let

- $\mathbf{x}$ be a point inside the convex set Ω;
- $Q(\mathbf{x}, \theta)$ be the **infinite** line through $\mathbf{x}$ in the direction θ.
- Let $\mathbf{p}_1(\mathbf{x}, \theta)$ and $\mathbf{p}_2(\mathbf{x}, \theta)$ be the two intersections between $Q(\mathbf{x}, \theta)$ and $\partial\Omega$,



then we define

$$g_{\mathrm{GW}}(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{lerp}\left(f(\mathbf{p}_1(\mathbf{x}, \theta)),\ f(\mathbf{p}_2(\mathbf{x}, \theta)),\ \frac{\|\mathbf{p}_1(\mathbf{x},\theta) - \mathbf{x}\|}{\|\mathbf{p}_1(\mathbf{x},\theta) - \mathbf{p}_2(\mathbf{x},\theta)\|} \right) d\theta.$$

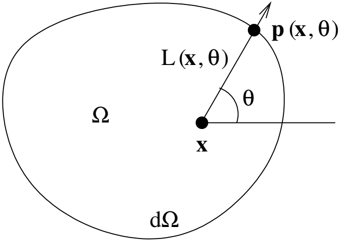
- Works only for convex sets.
- Evaluation requires numerical integration

$\Longrightarrow$ must find intersections for each integration point!

A mean value approach

Let

- $L(\mathbf{x}, \theta)$ be the semi-infinite line originating at $\mathbf{x}$ in the direction θ.
- $\mathbf{p}(\mathbf{x}, \theta)$ be the intersection of $L(\mathbf{x}, \theta)$ and $\partial\Omega$.



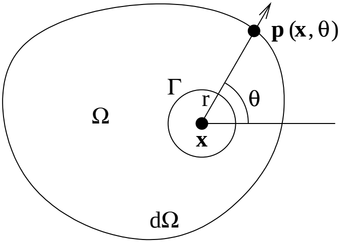
and define the "radially linear" function F as

- $F(\mathbf{x} + r(\cos\theta, \sin\theta)) = \text{lerp}\left(g(\mathbf{x}),\ f(\mathbf{p}(\mathbf{x}, \theta)),\ \frac{r}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|}\right).$

We want F to satisfy the Mean Value property at $\mathbf{x}$.

Let Γ be any circle at $\mathbf{x}$ with radius r, then

$$F(\mathbf{x}) = \frac{1}{2\pi r} \int_\Gamma F(\mathbf{z}) d\mathbf{z},$$



whose unique solution is

$$g(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})} \int_0^{2\pi} \frac{f(\mathbf{p}(\mathbf{x},\theta))}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|} d\theta, \quad \phi(\mathbf{x}) = \int_0^{2\pi} \frac{1}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|} d\theta,$$

which is the “angle integral” formula for the MV interpolant g.

- Generalizes to non-convex sets
- Evaluation still requires numerical integration.
- Still must find an intersection for each integration point!
- How do we differentiate this thing?
- Luckily, we have two other formulas. . .

The boundary integral formula [Ju, Schaefer, Warren 2005]

Suppose $\mathbf{c} : [a, b] \to \mathbb{R}^2$ is an anticlockwise representation of $\partial\Omega$.

Then

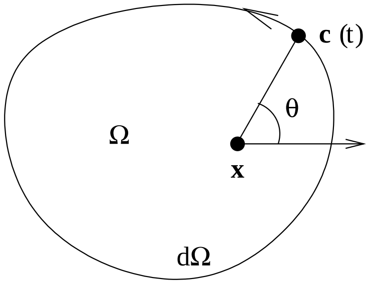
$$\frac{d\theta}{dt} = \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^2}$$

which gives

$$\phi(\mathbf{x}) = \int_a^b \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3} dt,$$

and

$$g(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})} \int_a^b \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3} f(\mathbf{c}(t)dt.$$



The polygonal formula

Suppose Ω is a polygon with vertices $\mathbf{p}_1, \ldots, \mathbf{p}_n$.

Then

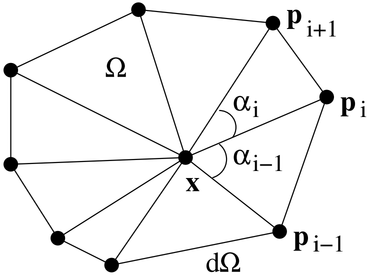
$$g(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})} \sum_i w_i(\mathbf{x}) f(\mathbf{p}_i),$$

where

$$\phi(\mathbf{x}) = \sum_i w_i(\mathbf{x}),$$

and

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}(\mathbf{x})/2) + \tan(\alpha_i(\mathbf{x})/2)}{\|\mathbf{p}_i - \mathbf{x}\|}.$$



We have three formulations for the MV interpolant:

- The polygonal formula:
  - closed form
  - easy to find expressions for derivatives.
- The boundary integral formula
  - needs adaptive numerical quadrature for evaluation.
  - easy to find expressions for derivatives.
- The angle integral
  - describes the interpolant along a particular direction.

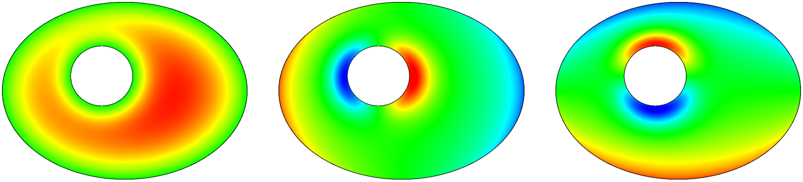
The MV weight function

A lot of properties can be deduced from the "weight function" ψ

$$\psi(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})} = 1 \Big/ \int_0^{2\pi} \frac{1}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|} d\theta,$$

$$= 1 \Big/ \int_a^b \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3} dt,$$

$$= 1 \Big/ \sum_i \frac{\tan(\alpha_{i-1}(\mathbf{x})/2) + \tan(\alpha_i(\mathbf{x})/2)}{\|\mathbf{p}_i - \mathbf{x}\|}.$$



ψ $\qquad$ $\frac{\partial\psi}{\partial x}$ $\qquad$ $\frac{\partial\psi}{\partial y}$

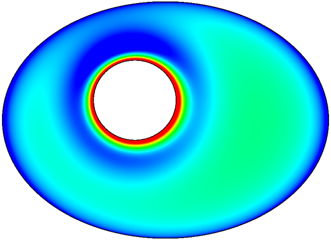
Minimum principle for ψ

For arbitrary Ω, we have that

$$\Delta\phi(\mathbf{x}) = 3\int_0^{2\pi} \sum_{j=1}^{n(\mathbf{x},\theta)} \frac{(-1)^{j-1}}{\|\mathbf{p}_j(x,\theta) - \mathbf{x}\|^3} d\theta$$

from which follows that

- ψ has no local minima in Ω.



$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$

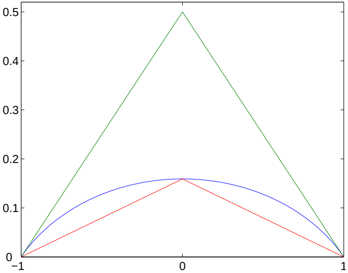
Bounds on ψ

For all $\mathbf{x} \in \Omega$ we have that

$$\frac{1}{2\pi}\,\mathrm{dist}(\mathbf{x}, \partial\Omega) \leq \psi(\mathbf{x}) \leq c\,\mathrm{dist}(\mathbf{x}, \partial\Omega),$$

- c depends on $\mathrm{dist}(M_E, \partial\Omega)$, the distance between $\partial\Omega$ and its exterior medial axis
- If Ω is convex, then $c = \frac{1}{2}$.

$\Longrightarrow$ For all $\mathbf{x} \in \Omega, \psi > 0$.



The plot shows the upper and lower bounds and ψ along a cross-section when Ω is the unit disc.

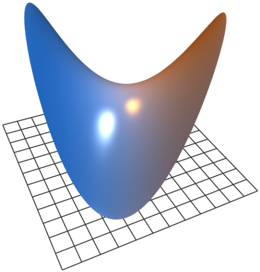
Proof of interpolation for the Lagrange MV interpolant

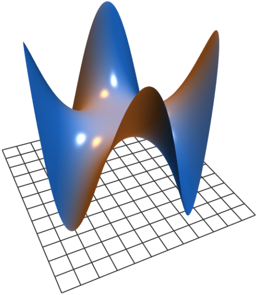
If

- f is continuous,
- $\partial\Omega$ and any line intersects a bounded number of times,
- and $\text{dist}(M_E, \partial\Omega) > 0$

then

- g interpolates f.





Normal derivatives of ψ and g

If

$$\text{dist}(M_E, \partial\Omega) > 0 \qquad \text{and} \qquad \text{dist}(M_I, \partial\Omega) > 0,$$

then, for all $\mathbf{y} \in \partial\Omega$,

- the inward normal derivative for ψ is

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{2}$$

- the inward normal derivate for the Lagrange interpolant g is

$$\frac{\partial g}{\partial \mathbf{n}}(\mathbf{y}) \;\; = \;\; \frac{1}{2}\int_a^b \frac{(\mathbf{c}(t) - \mathbf{y}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3}\big(f(\mathbf{c}(t)) \; - \; f(\mathbf{y})\big)\,dt.$$

Hermite mean value interpolation

In one variable, we have the problem

$$p(x_i) = f(x_i) \qquad \text{and} \qquad p'(x_i) = f'(x_i), \qquad i = 0, 1.$$

One approach of expressing p is

$$p(x) = g_0(x) + \psi(x) g_1(x),$$

where

- g_0 and g_1 are Lagrange interpolants,
- ψ vanishes at x_0 and x_1 and ψ' is nonzero at x_0 and x_1.

Which gives the conditions

$$g_0(x_i) = f(x_i) \qquad \text{and} \qquad g_1(x_i) = \frac{f'(x_i) - g_0'(x_i)}{\psi'(x_i)}.$$

In two variables, we can generalize a similar problem,

$$p(\mathbf{y}) = f(\mathbf{y}) \qquad \text{and} \qquad \frac{\partial p}{\partial \mathbf{n}}(\mathbf{y}) = \frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}), \qquad \mathbf{y} \in \partial\Omega.$$

and let p be on the form

$$p(\mathbf{x}) = g_0(\mathbf{x}) + \psi(\mathbf{x}) g_1(\mathbf{x}).$$

We can use the MV-ψ since

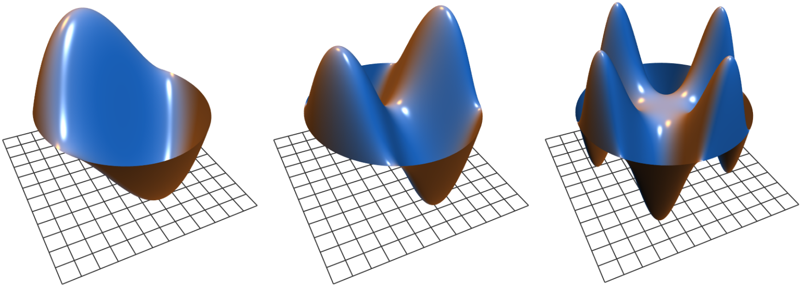
$$\psi(\mathbf{y}) = 0 \qquad \text{and} \qquad \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{2}, \qquad \mathbf{y} \in \partial\Omega,$$

and let g_0 and g_1 be MV Lagrange interpolants.

Then, for $\mathbf{y} \in \partial\Omega$ we get the conditions

$$g_0(\mathbf{y}) = f(\mathbf{y})$$

$$g_1(\mathbf{y}) = \left(\frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial g_0}{\partial \mathbf{n}}(\mathbf{y}) \right) \Bigg/ \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}).$$

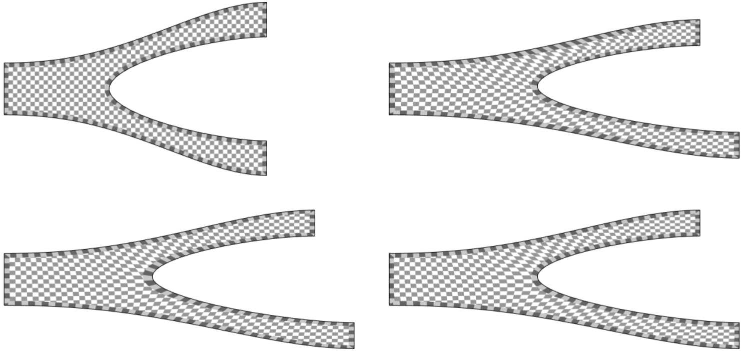


Application: Smooth mappings

Reference shape

Computational domain

(extended Gordon & Hall)



MV-Lagrange

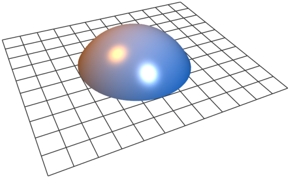
MV-Hermite

Conjecture: Lagrange interpolation from convex sets to convex sets is always injective.

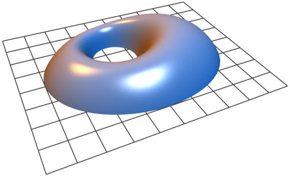
Application: WEB-splines [Höllig, Reif, Wipper 2001]

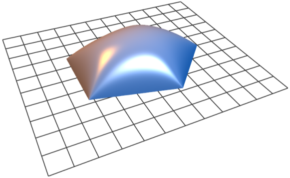
Idea: Use ψ as a weight function for WEB-splines

Parametric circle

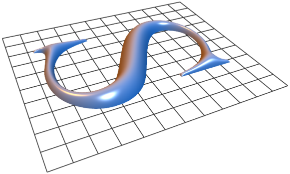


Two nested ellipses



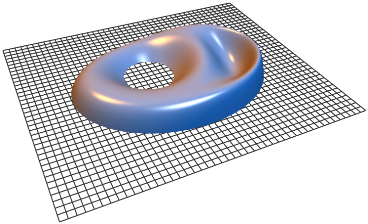


Polygon



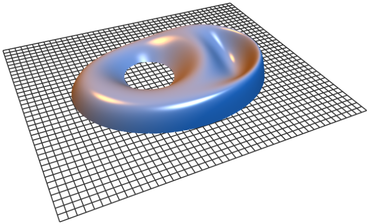
Piecwise cubic Bézier curve

Solution to Poisson's equation using bicubic web-splines



Using implicit weight function

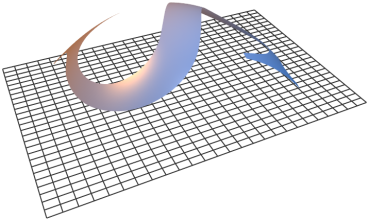
Grid	L2 error	order
10×8	7.3e-02	
20×16	2.9e-02	1.31
40×32	1.6e-03	4.21
80×64	4.4e-05	5.17



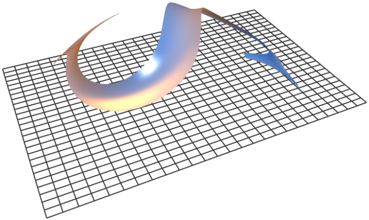
Using MV weight function

Grid	L2 error	order
10×8	9.5e-02	
20×16	4.1e-02	1.21
40×32	2.4e-03	4.12
80×64	1.4e-04	4.01

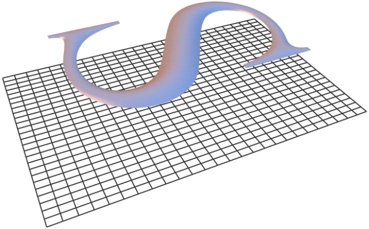
Inhomogenenous Poisson's equation



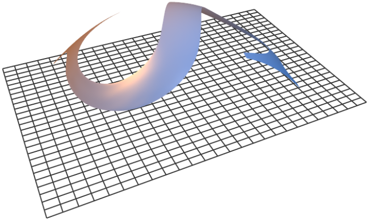
True solution



MV Lagrange interpolant



Homogeneous solution



Inhomogeneous solution

Conclusions

- The **Lagrange mean value interpolant** does in fact interpolate.
- Constructed a **Hermite mean value interpolant**.
- The **mean value weight function** has nice properties:
  - positive;
  - C^∞-smooth;
  - bounded by the distance function:

    $\Longrightarrow$ a very smooth distance-like function without ridges along the inner medial axis!
  - constant normal derivate;
  - has no local minima in Ω;
- The mean value constructions are relatively easy to compute:
  - The polygonal case has a closed form.
  - The boundary integral must be calculated numerically, but:
    - Strong influence of the boundary region closest to the point of evaluation.

    $\Longrightarrow$ Adaptivity pays off.
    - Simpler than solving a PDE.

Thank you for listening!

Questions?